Experiment 3: Frequency Modulation and Demodulation

Objective

Frequency modulation (FM) is a nonlinear modulation technique where the modulating signal varies the instantaneous frequency of the carrier wave. In this experiment you will generate FM signals and study their frequency-domain characteristics. You will examine the operational properties of a phase-locked loop (PLL) and use it to demodulate FM signals.

Prelab Assignment

1. Let $m(t) = A_m \cos 2\pi f_m t$ be the single-tone signal with $f_m = 1$ kHz and amplitude A_m. $m(t)$ generates the FM signal:

$$\varphi_{\text{FM}}(t) = A_c \cos\big(2\pi f_c t + K_f \int_0^t m(\lambda)d\lambda\big), \tag{1}$$

$$= A_c \cos\big(2\pi f_c t + \beta \sin 2\pi f_m t\big), \tag{2}$$

where $A_c = \sqrt{2}$ V, $f_c = 10$ kHz, $K_f = 3.45$ kHz/V is the frequency sensitivity parameter of the FM modulator and β is the modulation index:

$$\beta = \frac{K_f A_m}{2\pi f_m}. \tag{3}$$

(a) Determine values of the modulating signal amplitude A_m required generate FM signals with $\beta = 0.2, 1$ and 5.

(b) For each value of the modulation index β, i.e., for $\beta = 0.2, 1$ and 5, determine the rms values of the spectral components of the FM signal $\varphi_{\text{FM}}(t)$ using Table (A.1) in the Appendix. Also determine the rms value of the FM signal $\varphi_{\text{FM}}(t)$.

(c) For each value of the modulation index β, estimate the bandwidth of the FM signal $\varphi_{\text{FM}}(t)$ using:

i. Carson's rule;

ii. 1% rule;

iii. Universal curve corresponding to the 1-% rule.

2. Figure (1) shows the simplified block diagram of the PLL module based on the LM565C device. For further information on the PLL and application examples please refer to the technical specifications sheet posted on Blackboard.

The PLL module uses bias voltages $V_+ = 8$ V and $V_- = -8$ V. The timing resistor R_1 and the timing capacitor C_1 determine the VCO free-running frequency such that $f_0 \approx 0.3/(R_1 C_1)$. In this experiment, you will set $f_0 = 10$ kHz by adjusting R_1.

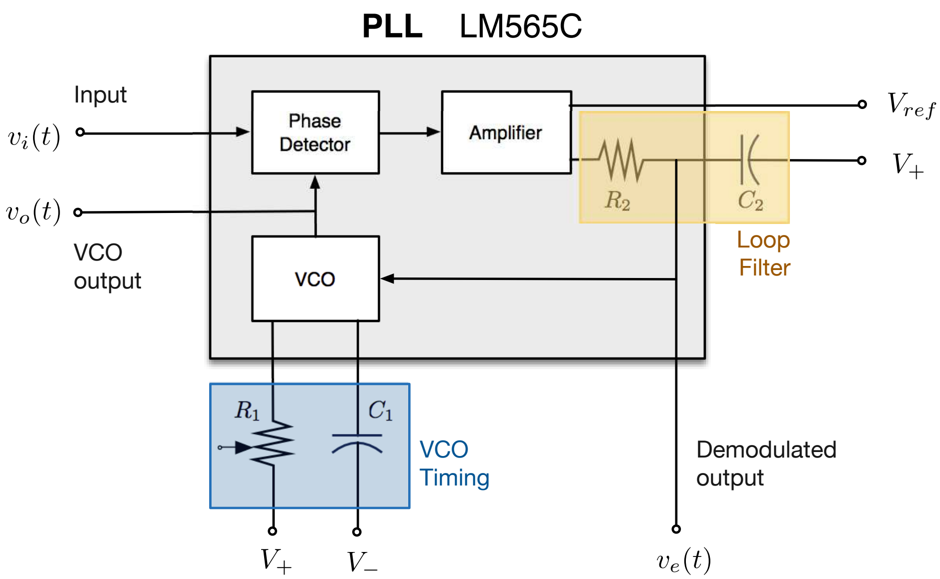

Figure 1: Simplified block diagram of the PLL module.

$R_2 = 3.6\ k\Omega$ appears on the chip as a portion of the loop filter. The external capacitor $C_2 = 0.047\ \mu F$ sets the time constant to $\tau = R_2C_2$ such that the -3 dB frequency of the loop filter becomes:

$$f_{lpf} = \frac{1}{2\pi\tau} = \frac{1}{2\pi R_2 C_2}. \tag{4}$$

The design equations associated with the PLL module are:

$$f_L \approx \pm \frac{8\, f_o}{V_+ + |V_-|}, \tag{5}$$

Lock (hold-in, tracking) **Range**

$$f_p \approx \pm\sqrt{f_L\ f_{lpf}}\,, \tag{6}$$

Pull-In (capture, acquisition) **Range**.

Determine:

(a) The lock range f_L;

(b) The pull-in range f_p.

3. Download to oscilloscope and spectrum analyzer setup files e3setupA.scp, e3setupC.scp, e3setupD2.scp and e3setupD3.scp from **[BlackBoard]** > **[Laboratory]** > **[Experiment 3]** to a USB drive.

Equipment

In this experiment you will use the following equipment and software:

- Agilent DSO-X 2002A digital storage oscilloscope with waveform generation and spectrum analyzer options.
- GW Instek GFG-8216A function generator.
- Hewlett Packard 33120A function/arbitrary waveform generator.
- PLL module based on LM565C chip.
- Lowpass filter module.
- Agilent E3630A triple output DC power supply.
- Computer with Linux operating system.
- Matlab/Simulink 2014b.

Procedure

Part-A/B Setup

Function Generator 1–HP 33120A **(FG1):** The settings are: **[Waveform:** sine **]**, **[Frequency:** 1 kHz**]**, **[Amplitude:** 2 V$_{pp}$**]**.

Function Generator 2–GFG-8216A **(FG2):** The settings are: **[Waveform:** sine**]**, **[Frequency:** 10 kHz**]** and **[Amplitude:** 2.83 V$_{pp}$**]**.

Oscilloscope/Spectrum Analyzer: Press **[Math]** > **[Operator:** FFT**]**. Use the following control settings: **[Source:** Channel 2 **]**, **[Span:** 20 kHz**]**, **[Center:** 10 kHz**]**, **[Window:** Rectangle**]**, **[Vertical Units:** V rms**]**. The last two control settings are accessible by pressing the **[More FFT]** softkey. **Preset:** Initial spectrum Analyzer settings used in **Part-A/B** are stored in the file e3setupA.scp. To use the preset values: Press **[Save/Recall]** > **[Recall]** > **[Load from:** e3setupA**]**.

A. Characteristics of FM Signals

Complete the connection diagram shown in Figure (2). Note that the output of FG1 must be connected to the **voltage controlled frequency (VCF)** input of FG2 located on its rear panel.

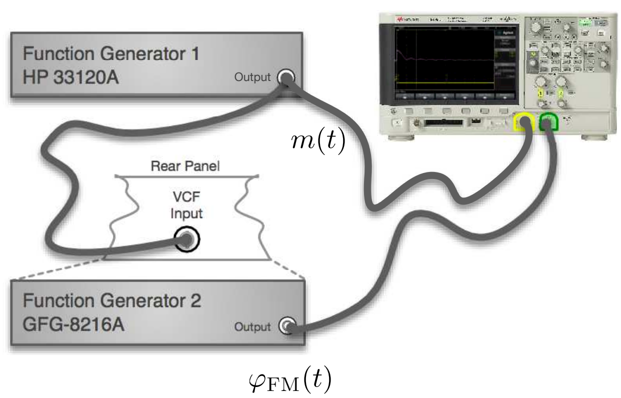

Figure 2: FM signal generation and **Part-A** connection diagram.

FG1 generates the single-tone modulating signal $m(t) = A_m \cos 2\pi f_m t$. With $m(t)$ connected to the VCF input, the output of FG2 becomes the FM signal $\varphi_{\mathrm{FM}}(t) = A_c \cos\big(2\pi f_c t + \beta \sin 2\pi f_m t\big)$, where β is the modulation index given in Equation (3).

Important : Use the peak-to-peak measurement of $m(t)$ as displayed on the oscilloscope to adjust the amplitude of the signal generated by FG1. The output amplitude of FG1 is calibrated to 50 Ω output load. Therefore, the amplitude display of FG1 will show a different value than what is measured by the oscilloscope. For example, when the amplitude display shows 1.0 V$_{pp}$, FG1 outputs a signal with amplitude of approximately 2.0 V$_{pp}$.

Step A.1 Connect the modulating signal $m(t)$ to Channel 1 and the FM signal $\varphi_{\mathrm{FM}}(t)$ to Channel 2 of the oscilloscope. Using the settings described in the **Part-A/B Setup**

section, display $m(t)$, $\varphi_{\mathrm{FM}}(t)$ and the one-sided rms spectrum of the FM signal on the oscilloscope. Use $m(t)$ as the trigger source.

Step A.2 Increase A_m, the amplitude of the modulating signal $m(t)$, from 2 V$_{pp}$ to 10 V$_{pp}$. Observe how the spectrum of the FM signal changes.

Step A.3 Reset A_m to 2 V$_{pp}$. Gradually reduce f_m, the frequency of the modulating signal $m(t)$, from 1 kHz to 100 Hz. Observe how the spectrum of the FM signal changes.

Step A.4 Set $A_m = 2\ V_{pp}$ and $f_m = 1$ kHz. Record the resulting one-sided rms spectrum of the FM signal. Change the modulating signal parameters to $A_m = 1\ V_{pp}$ and $f_m = 500$ Hz. Record the resulting one-sided rms spectrum of the FM signal.

Problem A.1 Based on your observations in **Step A.2–A.3** describe how the spectrum of a single-tone modulated FM signal changes as a function of the amplitude A_m and the frequency f_m of the modulating signal $m(t)$.

Problem A.2 Let $m(t) = A_m \sin \omega_m t$. Let $\Phi_{\mathrm{FM}}(f)$ be the one-sided rms spectrum of FM signal generated by $m(t)$. The modulating signal is changed to $m'(t) = KA_m \sin K\omega_m t$ where K is a positive integer. Let $\Phi'_{\mathrm{FM}}(f)$ be the one-sided rms spectrum of FM signal generated by $m'(t)$. Discuss how $\Phi_{\mathrm{FM}}(f)$ and $\Phi'_{\mathrm{FM}}(f)$ are related.

B. Modulation Index

Step B.1 Set $f_m = 1$ kHz and $A_m \approx 0\,V$. Gradually increase A_m until you observe a spectral null at carrier frequency $f_c = 10$ kHz. Record the value of A_m. Determine the modulation index β and the frequency sensitivity parameter K_f.

Step B.2 Continue increasing A_m until you observe a second spectral null at f_c. Record the value of A_m. Determine the modulation index β and the frequency sensitivity parameter K_f.

Step B.3 Let $f_m = 1$ kHz. Using the frequency sensitivity parameter $K_f = 3.45$ kHz/V, determine the A_m values to achieve modulation indices $\beta = 0.2$, 1 and 5.

Step B.4 Set the modulating signal amplitude to the value determined in **Step B.3** to achieve $\beta = 0.2$. Measure and record the magnitudes of spectral components of the one-sided rms spectrum of the FM signal. Ignore spectral components with magnitude values less than 10 mV.

Step B.5 Repeat **Step B.4** for $\beta = 1$.

Step B.6 Repeat **Step B.4** for $\beta = 5$.

Problem B.1 How you would calculate the modulation index β and the frequency sensitivity parameter K_f from the measurements taken in **Step B.1–B.2**.

Problem B.2 Let $[\varphi_{\text{FM}}]_{rms}$ be the rms value of the FM signal $\varphi_{\text{FM}}(t) = A_c \cos\big(\omega_c t + \beta \sin \omega_m t\big)$. Determine $[\varphi_{\text{FM}}]_{rms}$ directly and from the measurements you have taken in **Step B.4–B.6**.

Problem B.3 Estimate the bandwidth of the FM signal $\varphi_{\text{FM}}(t)$ with $\beta = 0.2, 1$ and 5, whose spectral characteristics measured in **Step B.4–B.6**, using: (i) Carson's rule and (ii) the 1% rule.

C. Operational Characteristics of Phase-Locked Loop

Part-C Setup

Function Generator 1–HP 33120A **(FG1):** The settings are: **[Waveform:** sine **]**, **[Frequency:** 1 kHz**]**, **[Amplitude:** 2 V_{pp}**]**.

Oscilloscope: Preset: Oscilloscope settings used in **Part-C** are stored in the file e3setupC.scp. To use the preset values: Press **[Save/Recall]** > **[Recall]** > **[Load from:** e3setupC**]**.

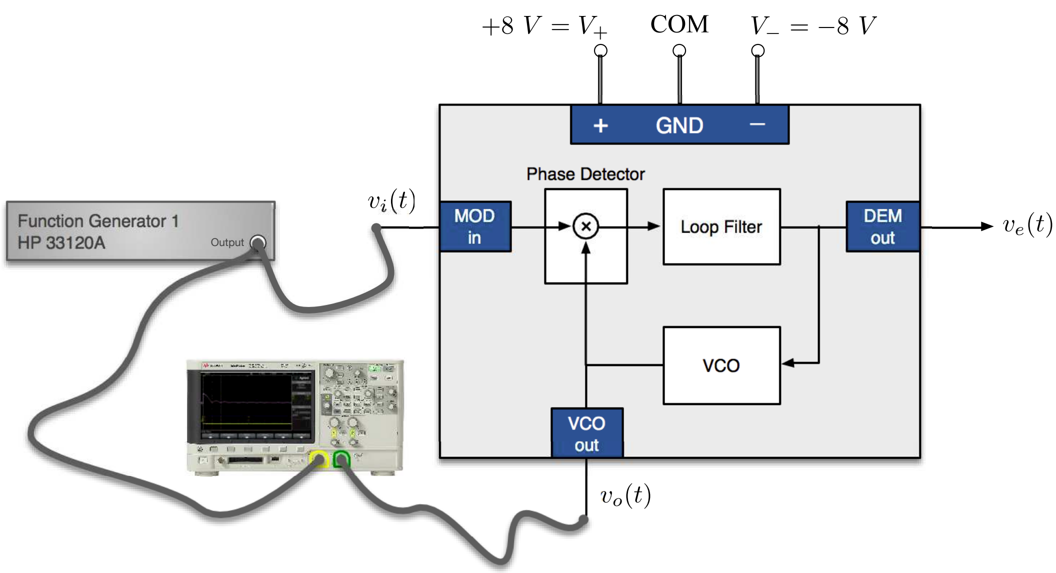

Figure 3: Connection diagram to measure PLL characteristics.

Step C.1 Complete the connection diagram shown in Figure (3). First, **disconnect** $v_i(t)$ from the **[MOD in]** terminal of the PLL module. Adjust the potentiometer (which corresponds to R_1 in Figure (1)) to set the free-running frequency of the VCO to 10 kHz. Use the DMM to measure the DC voltage V_e accessible through the **[DEM out]** terminal of the PLL module. The measured voltage is labeled as V_{ref} and represents the VCO input when the PLL is in *free-running* mode.

Step C.2 Reconnect FG1 to the **[MOD in]** terminal of the PLL module as shown in Figure (3). Adjust FG1 to generate $v_i(t) = A_i \cos 2\pi f_i t$ with $f_i = 4$ kHz and $A_i = 2\ V_{pp}$. Connect $v_i(t)$ to Channel 1 and $v_o(t)$ to Channel 2; use $v_i(t)$ as the trigger source. The traces will synchronize only when the PLL is in **lock condition**.

Gradually increase the input signal frequency f_i and determine the frequency f_{p^-} at which the traces suddenly synchronize, and remains in synch despite changes in the input frequency f_i. f_{p^-} defines the lower edge of the *pull-in* range of the PLL. The PLL is now in lock condition. Measure the DC voltage V_e at the **[DEM out]** terminal of the PLL module.

Step C.3 Increase the input frequency f_i in 1 kHz increments. While the PLL is **in lock condition**, for each f_i measure: (i) the DC value V_e at the **[DEM out]** terminal of the PLL module, and (ii) the phase angle of $v_o(t)$ with respect to $v_i(t)$. Determine the frequency at which the PLL can no longer track the input frequency f_i. f_{L^+} defines the upper edge of the *lock* range of the PLL.

Step C.4 Set $f_i = 16$ kHz. Gradually decrease the input frequency f_i, and determine f_{p^+} when $v_i(t)$ and $v_o(t)$ suddenly synchronize. f_{C^+} defines the upper edge of the *pull-in* range of the PLL. Decrease f_i in 1 kHz steps. While the PLL is **in lock condition**, for each f_i measure: (i) the DC value V_e at the **[DEM out]** terminal of the PLL module, and (ii) the phase angle of $v_o(t)$ with respect to $v_i(t)$. Determine the frequency at which the PLL can no longer track the input frequency f_i. f_{L^-} defines the lower edge of the *lock* range of the PLL.

Step C.5 This step as an automated version of the experimental procedure in **Step C.3**. Reconfigure FG2 by using the settings stored in memory location 2—on FG1 press **[Recall]**, select memory location **[2]** and press **[Enter]**. These settings place FG2 into a frequency sweep mode[1] with linear frequency spacing from $f_{start} = 4$ kHz to $f_{stop} = 17$ kHz and a sweep rate of 50 s. Connect $v_i(t)$ to Channel 1 and $v_o(t)$ to Channel 2; use $v_i(t)$ as the trigger source. Observe the behavior of the PLL as the input frequency f_i is swept from f_{start} to f_{stop}.

Problem C.1 Plot $V_e - V_{ref}$ as a function of f_i from the measurements taken in **Step C.2–C.3**, i.e., when f_i was increased from 4 kHz to beyond f_{L^+}. Also plot the phase angle between $v_0(t)$ and $v_i(t)$ as a function of f_i.

Problem C.2 Plot $V_e - V_{ref}$ and the phase angle between $v_0(t)$ and $v_i(t)$ as a function of f_i from the measurements taken in **Step C.4**, i.e., when f_i was decreased from 16 kHz to below f_{L^-}.

Problem C.3 In **Step C.2**, f_i was gradually increased from 4 kHz towards f_{p^-}. During this process, while $f_i < f_{p^-}$, there were few isolated narrow-band frequency positions when $v_o(t)$ and $v_i(t)$ synchronized. How do you explain this phenomenon?

[1] In the frequency sweep mode, the function generator outputs a sinusoid with either linear or logarithmic frequency spacing that "steps" from the start frequency f_{start} to the stop frequency f_{stop} at a user-specified defined sweep rate.

D. Demodulation of FM Signals

Part-D Setup

Function Generator 1–HP 33120A **(FG1):** The settings are: **[Waveform:** sine**]**, **[Frequency:** 100 Hz**]** and **[Amplitude:** 2 V_{pp}**]**.

Function Generator 2–GFG-8216A **(FG2):** The settings are: **[Waveform:** sine**]**, **[Frequency:** 10 kHz**]**, **[Amplitude:** 2 V_{pp}**]**.

Oscilloscope: Preset: Oscilloscope settings used in **Step D.2** are stored in the file e3setupD2.scp. Oscilloscope settings used in **Step D.3** are stored in the file e3setupD3.scp. To use the preset values: Press **[Save/Recall]** > **[Recall]** > **[Load from:** e3setupD2 or e3setupD3**]**.

Step D.1 The lowpass filter module smoothes the demodulated signal $v_e(t)$ at the **[DEM out]** terminal of the PLL module. Set the -3 dB frequency of lowpass filter at 1 kHz.

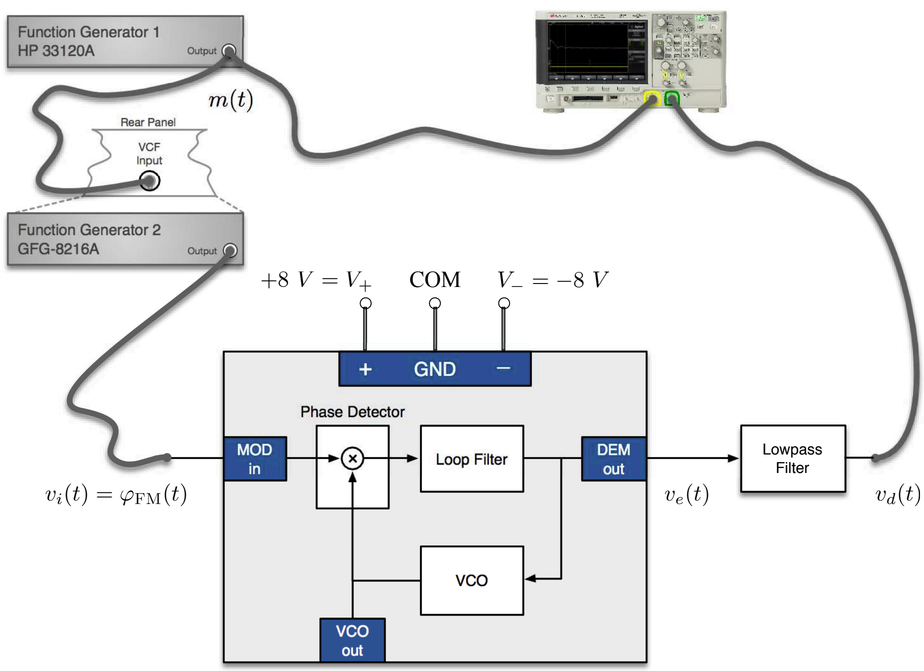

Figure 4: Connection diagram to demodulate FM signals.

Step D.2 Complete the connection diagram shown in Figure (4). Adjust FG1 to output a single-tone modulating signal $m(t)$ with frequency $f_m = 100$ Hz and amplitude 2 V_{pp}. (**Note:** Adjust the amplitude of $m(t)$ using the oscilloscope.)

Adjust FG2 to generate an FM signal with amplitude $A_c = 2\ V_{pp}$, carrier frequency $f_c = 10$ kHz. (**Note:** The amplitude setting of for the single-tone modulating signal $m(t)$ will result in an FM signal with modulation index $\beta = 5$.) Set the free-running frequency of the VCO to 10 kHz as before. Display the modulating signal $m(t)$ and the lowpass filtered demodulated signal $v_d(t)$ on the oscilloscope. Switch Channel 2 of the oscilloscope (to which $v_d(t)$ is connected) to **AC-coupled mode**.

Change the frequency of the modulating signal f_m and verify that the PLL can successfully extract the modulating signal $m(t)$ from $\varphi_{\text{FM}}(t)$ for $f_m < 1$ kHz.

Step D.3 Reconfigure the system by completing the connection diagram shown in Figure (5). In this setup. the modulating signal $m(t)$ is an audio file played back from the computer. Connect the **[Sound out port]** of the computer to the **VCF** terminal of FG1. Connect the demodulated signal $v_e(t)$ at the audio jack of the PLL module to the speaker attached to the LCD monitor.

Setup the oscilloscope to display the demodulated signal using the settings stored in file e3setupD3.scp. Open and run the Simulink model Exp3_PlayBack.slx.

Observe the modulating signal $m(t)$ on the Simulink virtual scope and the lowpass filtered demodulated signal $v_d(t)$ on the oscilloscope. You can verify the success of the PLL in demodulating the FM signal by listening to the demodulated signal $v_e(t)$ and/or by observing the displayed waveforms. (**Note:** The long timebase setting—2.0 s per division—causes noticeable latency in oscilloscope trace updates. However, this timebase setting is helpful in comparing the modulating signal $m(t)$ displayed on the Simulink virtual scope and the demodulated signal $v_d(t)$ displayed on the oscilloscope.)

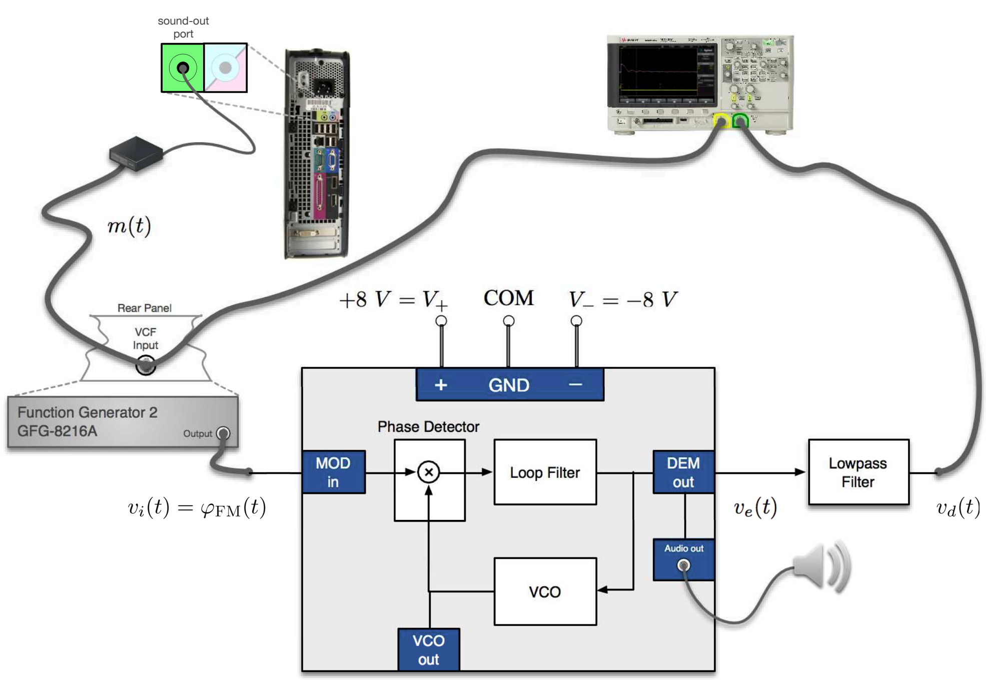

Figure 5: Connection diagram to the demodulate FM signals.

Step D.4 Connect the FM signal $\varphi_{\mathrm{FM}}(t)$ to the oscilloscope and display its spectrum $\Phi_{\mathrm{FM}}(f)$. For the spectrum analyzer, use **[Span:** 20 kHz**]**, **[Center:** 10 kHz**]**, the **[V rms]** settings. Observe the dynamic structure of $\Phi_{\mathrm{FM}}(f)$ and how the FM signal bandwidth changes as a function of the amplitude of the audio signal.

Extra-Curricular Problem Identify the artist(s) and the song in **Step D.3**.

Appendix: Bandwidth of Single-Tone Modulated FM Signals

Consider the single-tone modulating signal $m(t) = A_m \cos \omega_m t$. The corresponding FM signal can be expressed as:

$$\varphi_{\text{FM}}(t) = A_c \cos\big(\omega_c t + K_f \int_0^t m(\lambda) d\lambda\big) \tag{A.1}$$

$$= A_c \cos\big(\omega_c t + +\beta \sin \omega_m t\big), \tag{A.2}$$

$$= \textbf{Re}\Big\{A_c\, e^{j[\omega_c t + \beta \sin \omega_m t]}\Big\}, \tag{A.3}$$

$$= \textbf{Re}\Big\{A_c\, e^{j\beta \sin \omega_m t} e^{j\omega_c t}\Big\}, \tag{A.4}$$

where $\Delta f = K_f A_m/(2\pi)$ is the frequency deviation parameter and $\beta = \Delta f / f_m$ is the modulation index of the FM signal. The complex exponential function $e^{j\beta \sin \omega_m t}$ is periodic. Therefore, it can be expanded in the Fourier series:

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t} \tag{A.5}$$

with

$$C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt = J_n(\beta) \tag{A.6}$$

where $J_n(\beta)$ is the *Bessel function* of $1st$ kind, order n and argument β.

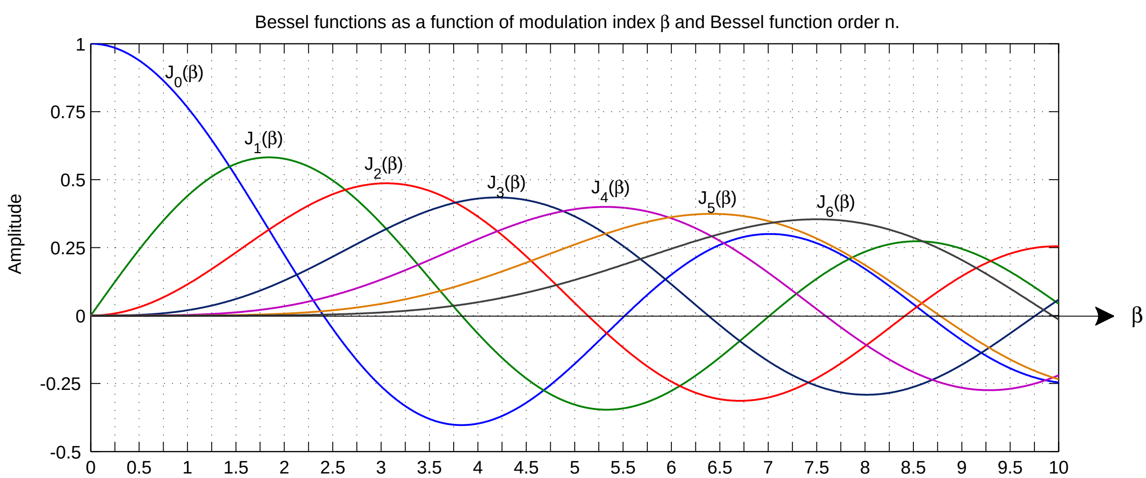

Figure A.1: Bessel functions as a function of modulation index β.

Using Equation (A.5) and the result in Equation (A.6), we can express $\varphi_{\text{FM}}(t)$ as:

$$\varphi_{\text{FM}}(t) = \mathbf{Re}\Big\{A_c\, e^{j\beta \sin\omega_m t}\, e^{j\omega_c t}\Big\} \tag{A.7}$$

$$= \mathbf{Re}\Big\{A_c\Big[\sum_n J_n(\beta) e^{jn\omega_m t}\Big] e^{j\omega_c t}\Big\} \tag{A.8}$$

$$= \mathbf{Re}\Big\{A_c \sum_n J_n(\beta) e^{j(\omega_c + n\omega_m)t}\Big\} \tag{A.9}$$

$$= A_c \sum_n J_n(\beta)\cos(\omega_c + n\omega_m)t. \tag{A.10}$$

The last expression is in a form suitable for computing the spectrum of the FM signal such that

$$\Phi_{\text{FM}}(f) = \mathcal{F}\big[\varphi_{\text{FM}}(t)\big] \tag{A.11}$$

$$= \mathcal{F}\big[A_c \sum_n J_n(\beta)\cos(\omega_c + n\omega_m)t\big] \tag{A.12}$$

$$= A_c \sum_n J_n(\beta)\,\mathcal{F}\big[\cos(\omega_c + n\omega_m)t\big] \tag{A.13}$$

$$= \frac{A_c}{2}\sum_n J_n(\beta)\,\big[\delta\big(f - (f_c + nf_m)\big) + \delta\big(f + (f_c + nf_m)\big)\big]. \tag{A.14}$$

Figure (A.2) shows the spectrum of a single-tone modulated FM signal.

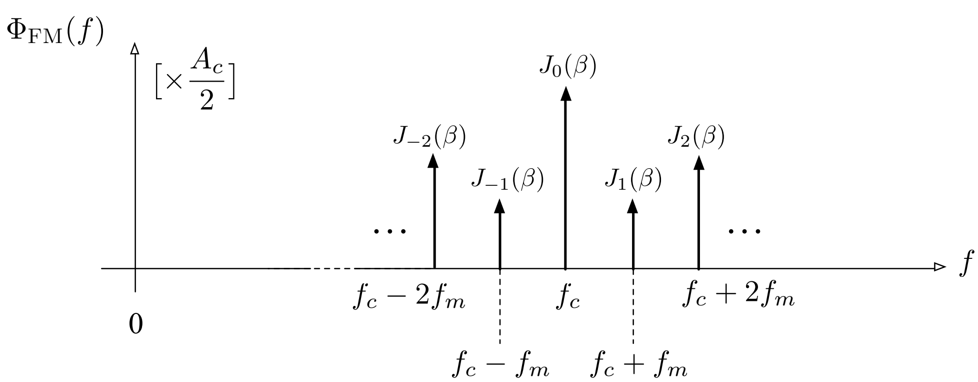

Figure A.2: Spectrum of a single-tonne modulated FM signal (shown for $f > 0$ only).

Estimating the Bandwidth of Single-Tone Modulated FM Signals

$\Phi_{\text{FM}}(f)$ has an infinite number of sidebands and therefore the FM signal $\varphi_{\text{FM}}(t)$ is not *strictly bandlimited*. The modulation index β determines $J_n(\beta)$ values, which in turn determine the shape of $\Phi_{\text{FM}}(f)$. In particular, we observe that:

- Magnitude of the spectral component at frequency $f_c + nf_m$ with $n \in \mathbb{Z}$ equals $A_c|J_n(\beta)|/2$;

- For fixed n, $|J_n(\beta)|$ decreases with increasing β;
- For fixed β, $|J_n(\beta)|$ decreases with increasing n, i.e., for $|f| \gg f_c$.

Therefore, we conclude that the power of $\varphi_{\text{FM}}(t)$ is contained within a *finite bandwidth*.

Narrowband FM (NBFM): $\beta \leq 0.3$

In the NBFM case, Figure (A.1) shows that $J_0(\beta) \approx 1$, $J_1(\beta) = -J_{-1} = \beta/2$ and $|J_n(\beta)| \approx 0$ for $|n| \geq 2$. Consequently, the expression for the FM signal $\varphi_{\text{FM}}(t)$ in Equation (A.10) reduces to:

$$\varphi_{\text{FM}}(t) \approx A_c \cos \omega_c t + A_c\, J_1(\beta) \cos(\omega_c + \omega_m)t + A_c\, J_{-1}(\beta) \cos(\omega_c - \omega_m)t \tag{A.15}$$

Therefore, the bandwidth of the NBFM signal becomes:

$$B_{\text{FM}} \approx 2f_m. \tag{A.16}$$

Wideband FM (WBFM): $\beta > 0.3$

In the WBFM case, we need to identify how many sidebands are considered *significant* to be included in calculating the bandwidth of the FM signal. A common rule is that a sideband is significant if its magnitude exceeds x-% of the magnitude of the unmodulated carrier. Most commonly used values are 1% (the 1-% rule) and 10% (the 10-% rule).

From Equation (A.14) we observe that the magnitude of the unmodulated carrier is $A_c J_0(0)$ whereas the magnitude of a sideband located at $f_c \pm n f_m$ is $A_c|J_n(\beta)|$. Under the 1-% rule we determine the sideband index n such that:

$$A_c|J_n(\beta)| > 10^{-2} A_c J_0(0). \tag{A.17}$$

As $J_0(0) = 1$, the condition expressed in Equation (A.17) is equivalent to

$$|J_n(\beta)| > 10^{-2}. \tag{A.18}$$

Let $n_{\max}$ be the largest value of the sideband index satisfying the requirement $|J_n(\beta)| > 10^{-2}$. Typically, we determine $n_{\max}$ from a table of values of Bessel function as shown in Table (A.1). Once we determine the sideband index $n_{\max}$, then we estimate the bandwidth of the FM signal as:

$$B_{\text{FM}} \approx 2\, n_{\max} f_m. \tag{A.19}$$

Alternatively, the x-% rule can also be presented in the form of a *universal curve* by normalizing the results obtained from the Bessel function tables with respect to the frequency deviation Δf and then plotting it as a function of β. Figure (A.3) shows the *universal curve* corresponding to the 1-% rule. We then estimate the bandwidth B_{FM} using the modulation index β and the frequency deviation Δf parameters.

Table of Bessel Functions as a function of modulation index β and Bessel function order n.

	β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$	$J_6(\beta)$	$J_7(\beta)$	$J_8(\beta)$	$J_9(\beta)$
β	0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.2	0.99	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.4	0.96	0.20	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.6	0.91	0.29	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.8	0.85	0.37	0.08	0.01	0.00	0.00	0.00	0.00	0.00	0.00
	1.0	0.77	0.44	0.11	0.02	0.00	0.00	0.00	0.00	0.00	0.00
	1.2	0.67	0.50	0.16	0.03	0.01	0.00	0.00	0.00	0.00	0.00
	1.4	0.57	0.54	0.21	0.05	0.01	0.00	0.00	0.00	0.00	0.00
	1.6	0.46	0.57	0.26	0.07	0.01	0.00	0.00	0.00	0.00	0.00
	1.8	0.34	0.58	0.31	0.10	0.02	0.00	0.00	0.00	0.00	0.00
	2.0	0.22	0.58	0.35	0.13	0.03	0.01	0.00	0.00	0.00	0.00
	2.2	0.11	0.56	0.40	0.16	0.05	0.01	0.00	0.00	0.00	0.00
	2.4	0.00	0.52	0.43	0.20	0.06	0.02	0.00	0.00	0.00	0.00
	2.6	-0.10	0.47	0.46	0.24	0.08	0.02	0.01	0.00	0.00	0.00
	2.8	-0.19	0.41	0.48	0.27	0.11	0.03	0.01	0.00	0.00	0.00
	3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01	0.00	0.00	0.00
	3.2	-0.32	0.26	0.48	0.34	0.16	0.06	0.02	0.00	0.00	0.00
	3.4	-0.36	0.18	0.47	0.37	0.19	0.07	0.02	0.01	0.00	0.00
	3.6	-0.39	0.10	0.44	0.40	0.22	0.09	0.03	0.01	0.00	0.00
	3.8	-0.40	0.01	0.41	0.42	0.25	0.11	0.04	0.01	0.00	0.00
	4.0	-0.40	-0.07	0.36	0.43	0.28	0.13	0.05	0.02	0.00	0.00
	4.2	-0.38	-0.14	0.31	0.43	0.31	0.16	0.06	0.02	0.01	0.00
	4.4	-0.34	-0.20	0.25	0.43	0.34	0.18	0.08	0.03	0.01	0.00
	4.6	-0.30	-0.26	0.18	0.42	0.36	0.21	0.09	0.03	0.01	0.00
	4.8	-0.24	-0.30	0.12	0.40	0.38	0.23	0.11	0.04	0.01	0.00
	5.0	-0.18	-0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	0.01
	5.2	-0.11	-0.34	-0.02	0.33	0.40	0.29	0.15	0.07	0.02	0.01
	5.4	-0.04	-0.35	-0.09	0.28	0.40	0.31	0.18	0.08	0.03	0.01
	5.6	0.00	-0.33	-0.15	0.23	0.39	0.33	0.20	0.09	0.04	0.01
	5.8	0.09	-0.31	-0.20	0.17	0.38	0.35	0.22	0.11	0.05	0.02
	6.0	0.15	-0.28	-0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02

Table A.1: Bessel functions as a function of modulation index β and Bessel function order n.

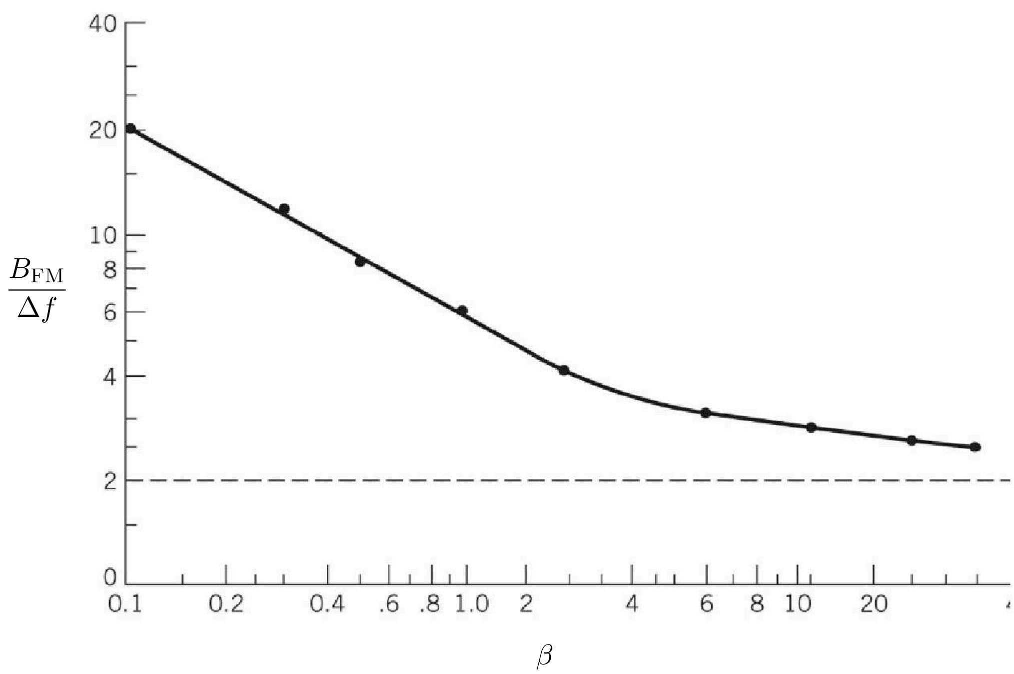

Figure A.3: Universal curve for the 1-% rule.